

CUTTING ARCS FOR TORUS LINKS AND TREES

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ABSTRACT. Among all torus links, we characterise those arising as links of simple plane curve singularities by the property that their fibre surfaces admit only a finite number of cutting arcs that preserve fibredness. The same property allows a characterisation of Coxeter-Dynkin trees (i.e., A_n , D_n , E_6 , E_7 and E_8) among all positive tree-like Hopf plumbings.

1. INTRODUCTION

A *fibred link* is a link $L \subset S^3$ such that $S^3 \setminus L$ fibers over the circle, and where each fibre is the interior of a Seifert surface S for L in S^3 . Cutting S along a properly embedded interval α (an *arc* for short) results in another Seifert surface S' for another link $\partial S' = L'$. If L' is again a fibred link with fibre S' , we say that α *preserves fibredness*. For example, α could be the spanning arc of a plumbed Hopf band, and cutting along α amounts to deplumbing that Hopf band. In [BIRS], Buck et al. give a simple criterion for when an arc preserves fibredness in terms of the monodromy $\varphi: S \rightarrow S$. As a corollary, they prove that each of the torus links of type $T(2, n)$ admits only a finite number of such arcs up to isotopy. It turns out that among torus links, this is an exception:

Theorem 1. *Let $n, m \geq 4$ or $n = 3, m \geq 6$. Then the fibre surface S of the torus link $T(n, m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.*

The remaining torus links $T(2, n)$, $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$ happen to be exactly those torus links that can also be obtained as plumbings of positive Hopf bands according to a finite tree, where vertices correspond to positive Hopf bands and edges indicate plumbing.

Theorem 2. *Let S be the fibre surface obtained by plumbing positive Hopf bands according to a finite tree T . There are, up to isotopy, only finitely many cutting arcs in S preserving fibredness, if and only if T is one of the Coxeter-Dynkin trees A_n , D_n , E_6 , E_7 or E_8 .*

To prove the “only if” part of Theorem 2, we consider orbits of a fixed arc under the monodromy to produce families of arcs that preserve fibredness. The basic idea is that such an orbit is infinite if the monodromy has infinite order. For example, we show that in fact every (prime) positive braid link with pseudo-Anosov monodromy admits infinitely many non-isotopic arcs preserving fibredness. This suggests the following question: is it true that among all (non-split prime) positive braid links, the ADE plane curve singularities are exactly those that admit just a finite number of fibredness preserving arcs up to isotopy?

Plan of the article. We use the shorthand *ADE links* to refer to the links of the positive tree-like Hopf plumbings according to the trees A_n , D_n , E_6 , E_7 or E_8 . The subsequent section combines a criterion on arcs to preserve fibredness from [BIRS] with the property of monodromies of positive Hopf plumbed surfaces to be right-veering. This allows for the following simple test for an arc to preserve fibredness, in our situation: an arc preserves fibredness if and only if it does not intersect its image under the monodromy (up to free isotopy).

Section 3 contains descriptions of the fibre surfaces and the monodromies of the links we consider (torus links and the *ADE* links). Alongside, we give a constructive proof of Theorem 1.

In Section 4, we explain the idea of proof for the finiteness result that provides the “if” part of Theorem 2, and list the fibred links obtained by cutting the fibre surfaces of the *ADE* links along an arc in Table 1.

Section 5 accounts for the cases where the monodromy has infinite order. This concerns in particular the positive tree-like Hopf plumbings that correspond to trees different from the *ADE* trees and settles the “only if” part of Theorem 2.

At the beginning of Section 6, we set up the notation and methods needed for the proof of the finiteness part of Theorem 2, which we split into Proposition 1 (concerning torus links) and Proposition 2 (concerning tree-like Hopf plumbings). The rest of that section is devoted to the proofs of these propositions.

2. RIGHT-VEERING SURFACE DIFFEOMORPHISMS AND CUTTING ARCS THAT PRESERVE FIBREDNESS

In the sequel we would like to make statements on the relative position of two arcs α, β in a surface S with boundary (that is, α, β are embedded intervals with endpoints on the boundary of S that are nowhere tangent to ∂S). The following definition will simplify matters.

The remainder of this section will recall the fact that every positive braid link (that is, the closure of a braid word consisting only of the positive generators of the braid group, without their inverses) is fibred and has so-called *right-veering* monodromy (see below for a definition). The torus links $T(n, m)$ provide examples, since they can be viewed as the closures of the positive braids $(\sigma_1 \cdots \sigma_{n-1})^m$, where the σ_i denote the (positive) standard generators of the braid group.

It is known that every positive braid can be obtained as an iterated plumbing of positive Hopf bands (see [St]). Since a Hopf band is a fibre and plumbing preserves fibredness, every positive braid link is fibred. Moreover the monodromy is a product of positive Dehn twists, since the monodromy of a (positive) Hopf band is a (positive) Dehn twist and the monodromy of a plumbing is the composition of the monodromies of the plumbed surfaces (see [Ga]). A product of positive Dehn twists is right-veering [HKM]. So we conclude that every positive braid link is fibred with right-veering monodromy. Together with a theorem by Buck et al., this property implies the following simple geometric criterion for when an arc preserves fibredness.



Theorem 3 (compare Theorem 1 in [BIRS]). *Let L be a fibred link with fibre surface S and right-veering monodromy $\varphi : S \rightarrow S$. Then, a cutting arc α preserves fibredness if and only if $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies on α and $\varphi(\alpha)$.*

Proof. This is a special case of Theorem 1 in [BIRS], saying that the arc α preserves fibredness if and only if α is *clean and alternating* or *once unclean and non-alternating* (see Figure 1), without the assumption on φ to be right-veering. But for a right-veering map, every arc is alternating, by definition. Finally, α is clean if and only if $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies on α and $\varphi(\alpha)$. \square

Remark 1. An arc α is clean if and only if $\varphi^k(\alpha)$ is clean, for all $k \in \mathbb{Z}$. This is clear since $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies if and only if $\varphi^k(\alpha) \cap \varphi^{k+1}(\alpha) = \partial\alpha$ after minimising isotopies. Similarly, if $\tau : S \rightarrow S$ is a homeomorphism such that $\varphi \circ \tau \circ \varphi = \tau$, then α is a clean arc if and only if $\alpha' = \tau(\varphi(\alpha))$ is. Indeed, $\alpha \cap \varphi(\alpha) = \partial\alpha \Leftrightarrow \tau(\alpha) \cap \tau(\varphi(\alpha)) = \partial\alpha' \Leftrightarrow \varphi(\tau(\varphi(\alpha))) \cap \tau(\varphi(\alpha)) = \partial\alpha' \Leftrightarrow \varphi(\alpha') \cap \alpha' = \partial\alpha'$.

3. MONODROMY OF TORUS LINKS, E_7 AND D_n .

The links that correspond to the trees A_n , E_6 and E_8 are torus links, namely $A_{n-1} = T(2, n)$, $E_6 = T(3, 4)$ and $E_8 = T(3, 5)$. Together with D_4 , which is $T(3, 3)$, these form the intersection between torus links and positive tree-like Hopf plumbings. For our purpose it therefore suffices to study torus links, E_7 and the D_n family.

The monodromies $\varphi : S \rightarrow S$ of the links in question are particular examples of *tête-à-tête twists*, a notion invented by A'Campo and further developed by Graf in his thesis [Gr]. This means that there exists a φ -invariant spine $\Gamma \subset S$, called *tête-à-tête graph*. Cutting S along the tête-à-tête graph results in finitely many annuli, on which φ descends to certain twist maps. More precisely, each of these annuli has one component of ∂S as one boundary circle and a cycle consisting of edges of Γ as the other. φ fixes ∂S pointwise and rotates the edge-cycles by some number ℓ of edges. The number $\ell \in \mathbb{Z}$ is called the *twist length* of the corresponding boundary annulus. After an isotopy (fixing the boundary of S), we may therefore assume that φ is periodic except on some annular neighbourhoods of ∂S . It is thus easy to understand the effect of φ on an arc α , up to isotopy, given the combinatorics of the action of φ on Γ and the amount of twisting on each annulus. Note that tête-à-tête twists define periodic mapping classes in the sense that some power is freely isotopic to the identity. However, this isotopy cannot be taken to be fixed on the boundary of S .

In a way dual to the tête-à-tête graph, we will find in each case a finite set of disjoint arcs that are permuted by φ and which decompose S into finitely many polygons, one for each vertex of Γ . The combinatorics of how these polygons are permuted will be used to prove Theorems 1 and 2.

Monodromy of torus links. The fibre surface S of the torus link $T(n, m)$ can be constructed as thickening of a complete bipartite graph on n and m vertices in the following way, as in Figure 2. Arrange

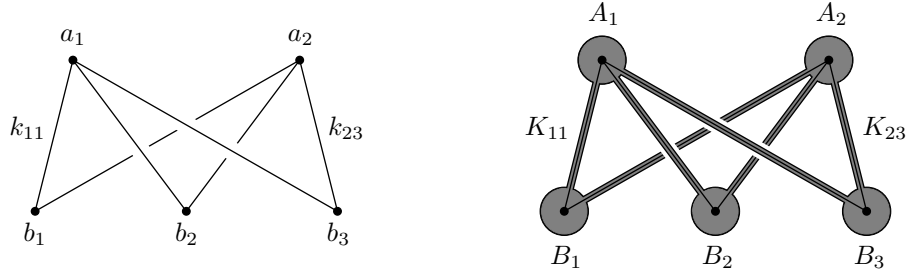


FIGURE 2. The complete bipartite graph on 2 and 3 vertices and blackboard framed thickening.

n collinear points a_1, \dots, a_n (in this order) in a plane and, similarly, another m points b_1, \dots, b_m along a line parallel to the a_i . Connect a_i and b_j by a straight segment k_{ij} , for every $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$. Avoid intersections between the segments by letting k_{ij} pass slightly under k_{pq} if $i > p$ and $j < q$ (in a slight thickening of the plane containing the points a_i and b_j). Use the blackboard-framing to thicken a_i, b_j, k_{ij} to disks A_i, B_j and bands K_{ij} . Choose the thickness of the bands K_{ij} so that they do not intersect outside the disks A_i, B_j . It can be seen that $S := \bigcup_i A_i \cup \bigcup_j B_j \cup \bigcup_{i,j} K_{ij} \subset \mathbb{R}^3 \subset S^3$ is isotopic to the minimal Seifert surface of $T(n, m)$ in S^3 (compare [Ba]). In addition, the monodromy $\varphi : S \rightarrow S$ is a tête-à-tête twist along the above graph. In each of the $\gcd(n, m)$ complementary annuli, φ fixes ∂S pointwise and rotates the edge-cycles two edges to the right with respect to the orientation of S . Using this description, it is possible to see that φ acts on the graph as follows: $\varphi(a_i) = a_{i-1}$, $\varphi(b_j) = b_{j+1}$, $\varphi(k_{ij}) = k_{i-1, j+1}$, where the indices i, j are to be taken modulo n, m respectively. A subarc of α that travels near k_{ij} will be mapped to a subarc of $\varphi(\alpha)$ that travels near $k_{i-1, j+1}$. The edges k_{ij} induce a decomposition of ∂A_i into circular arcs lying between points of the form $k_{ij} \cap \partial A_i$ (and the same for B_j). If $n, m \geq 3$, it is hence meaningful to speak of points on ∂A_i between k_{ij} and $k_{i, j+1}$.

Theorem 1. *Let $n, m \geq 4$ or $n = 3, m \geq 6$. Then the fibre surface S of the torus link $T(n, m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.*

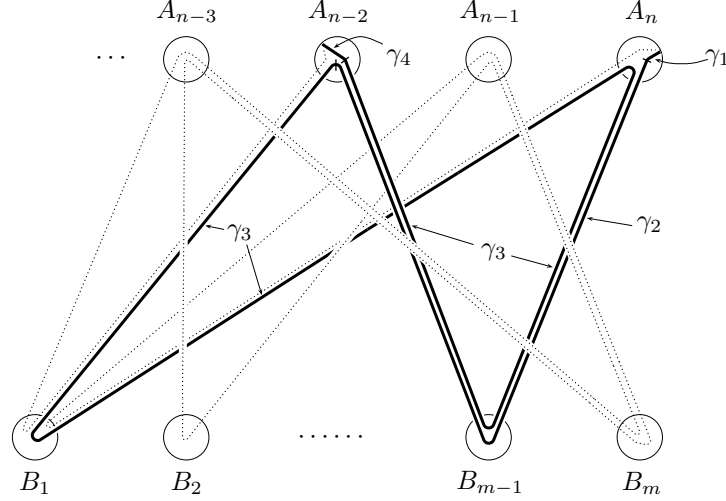


FIGURE 3. The arc $\alpha_1 = \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$ (solid line) and its image under the monodromy (dotted line). Note that these two arcs do not intersect, except at their endpoints.

Proof. For $n, m \geq 4$ consider the following arcs in S , using the notation from above (compare Figure 3):

- Let γ_1 be a straight segment starting at a point of ∂A_n between k_{n1} and k_{nm} and ending at the vertex a_n .
- Let γ_2 start at a_n , follow the edges $k_{n,m-1}$ and $k_{n-2,m-1}$, thus ending at a_{n-2} .
- γ_3 starts at a_{n-2} , runs along $k_{n-2,1}$, k_{n1} , $k_{n,m-1}$, $k_{n-2,m-1}$ and ends again at a_{n-2} .
- γ_4 is a straight segment from a_{n-2} to a point of ∂A_{n-2} between $k_{n-2,1}$ and $k_{n-2,m}$.

From $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ we can build an infinite family $(\alpha_r)_{r \in \mathbb{N}}$ of arcs in S , taking $\alpha_r = \gamma_1 * \gamma_2 * \underbrace{\gamma_3 * \dots * \gamma_3}_{r\text{-times}} * \gamma_4$. Here, $*$ denotes concatenation of

paths. Replacing the r consecutive copies of γ_3 by r parallel copies, the α_r can be thought of as embedded arcs. It is now easy to check that α_r and its image $\varphi_* \alpha_r$ under the monodromy φ have only their endpoints in common. Using Theorem 3 it follows that each α_r preserves fibredness. Finally, the α_r are homologically pairwise distinct. This can be seen in the following way: Let $[c] \in H_1(S, \mathbb{Z})$ be the cycle represented by

a simple closed curve c whose image is $k_{nm} \cup k_{n-1,m} \cup k_{n-1,m-1} \cup k_{n,m-1}$. After an isotopy, c will intersect α_r transversely in $r + 1$ points. Now, the linear form on $H_1(S, \partial S, \mathbb{Z})$ that sends α to $i(c, \alpha)$, the number of intersections with c (counted with signs), defines an element c^* of $H^1(S, \partial S, \mathbb{Z})$ such that $c^*(\alpha_r) = r + 1$, hence the claim.

If $n = 3, m \geq 6$, take the following arcs (compare Figure 4):

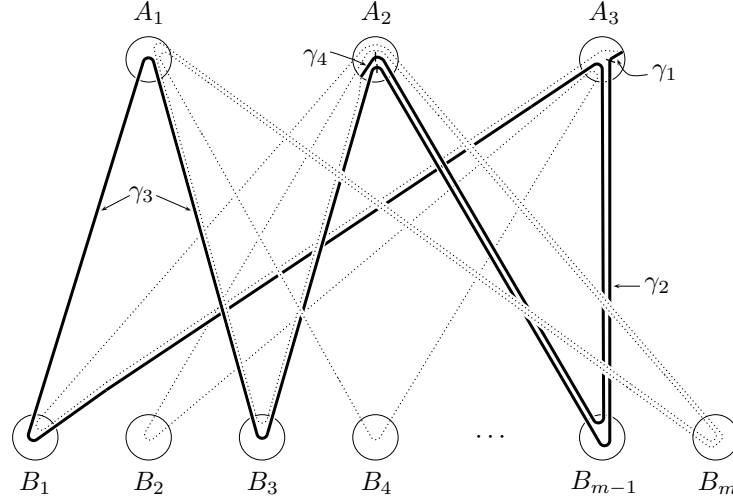


FIGURE 4. The arc α_1 (solid line) and its image under the monodromy (dotted line) for a $T(3, m)$ torus link, $m \geq 6$. Again, the two arcs do not intersect.

- γ_1 is a straight segment from a point of ∂A_3 between k_{31} and k_{3m} to a_3 .
- γ_2 starts at a_3 , follows the edges $k_{3,m-1}$ and $k_{2,m-1}$, thus ending at a_2 .
- γ_3 starts at a_2 , follows k_{23} , k_{13} , k_{11} , k_{31} , $k_{3,m-1}$ and $k_{2,m-1}$, ending at a_2 .
- γ_4 is a straight segment from a_2 to a point of ∂A_2 between k_{22} and k_{23} .

As above, we get a family $(\alpha_r)_{r \in \mathbb{N}}$ of homologically distinct arcs preserving fibredness, where $\alpha_r = \gamma_1 * \gamma_2 * \underbrace{\gamma_3 * \dots * \gamma_3}_{r\text{-times}} * \gamma_4$, using the curve with image $k_{3m} \cup k_{1m} \cup k_{1,m-2} \cup k_{3,m-2}$ to distinguish the α_r . \square

Monodromy of E_7 and D_n . In order to obtain a similar model for the fibre surface S of E_7 or D_n , start with two disjoint planar disks D, D' in \mathbb{R}^3 and connect them by half twisted bands b_1, \dots, b_n , where $n = 7$ in the case of E_7 . The embedded surface $S' = D \cup D' \cup b_1 \cup \dots \cup b_n$

is then a fibre surface for the $T(2, n)$ torus link. Let $p \in \partial D$ be a point between b_2 and b_3 in the case of D_n , respectively between b_3 and b_4 in the case of E_7 . Let I be an arc in D from a point of ∂D between b_1 and b_n to p . Finally, define S to be the surface obtained from S' by plumbing a positive Hopf band along I below the surface S' . Denote the core curve of that plumbed Hopf band by e_1 (so $e_1 \cap S' = I$). Each pair of consecutive bands b_i, b_{i+1} , $1 \leq i \leq n$, gives rise to a closed curve e_{i+1} that runs from D to D' through b_i and back to D through b_{i+1} .

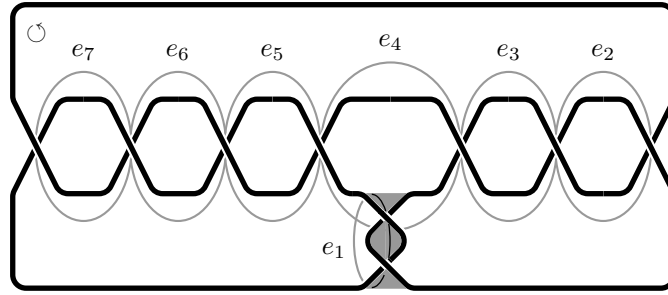


FIGURE 5. E_7 fibre surface with homology basis coming from the plumbing tree. φ is the product of the right handed Dehn twists on the curves e_i .

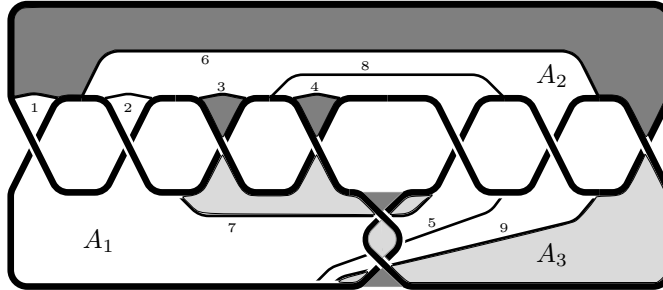


FIGURE 6. Decomposition of the surface into three hexagons A_1, A_2, A_3 . Hexagon A_3 is shaded grey. The monodromy permutes the intervals k_i (marked $1, 2, \dots, 9$) cyclically.

The incidence graph for the system of curves e_1, \dots, e_n in S is exactly the respective Coxeter-Dynkin tree E_7 or D_n (compare Figure 5). The

e_i are core curves of positive Hopf bands and S is a tree-like positive Hopf plumbing according to the respective tree. In particular, the monodromy φ of S is the product of the right handed Dehn twists about the curves $e_2, e_3, \dots, e_n, e_1$, in this order. Just as in the case of torus links, we will find a finite number of disjoint arcs in S that are permuted (up to free isotopy) by φ and such that these arcs cut S into polygons. For E_7 , let k_1 be the spanning arc of b_7 , and let $k_{i+1} = \varphi^i(k_1)$, $i = 1, \dots, 8$, up to free isotopy (compare Figure 6). Up to free isotopy, $\varphi(k_9) = k_1$. This can be seen by applying the Dehn twists about the e_j to the k_i , as described above. Another more visual way to see this is via *dragging arcs*. Imagine the arcs k_i to be elastic bands whose ends are attached to the surface boundary and whose interiors are pushed slightly off the surface into the positive normal direction. Applying the monodromy φ amounts to dragging the arc through the complement of S to the negative side of the surface, while its endpoints stay fixed on ∂S . Since we are only interested in the position of $\varphi(k_i)$ up to free isotopy, the endpoints of the dragging arc may move freely along ∂S during that process. Let A_1, A_2, A_3 be the three disk components of $S \setminus \bigcup_{i=1}^9 k_i$. The boundary of A_j alternates between parts of ∂S and the k_i . We choose the order as in Figure 7, where the components of $\partial A_j \cap \partial S$ are shrunk to points.

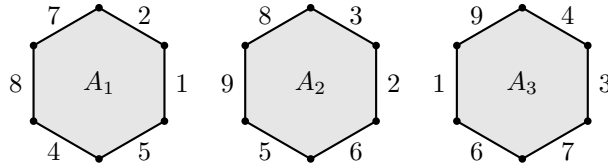


FIGURE 7. Edges with the same label are glued. The monodromy sends A_j to A_{j+1} , indices taken modulo 3, such that edge k_i is sent to edge k_{i+1} , modulo 9.

Examination of the action of φ on the k_i reveals that the A_i are cyclically permuted by φ , in the order $A_1 \mapsto A_2 \mapsto A_3 \mapsto A_1$. In Figure 7, the A_i are drawn in such a way that $A_1 \mapsto A_2 \mapsto A_3$ by translation to the right, and A_3 is mapped to A_1 by a translation, followed by a clockwise rotation through $1/3$. To obtain the tête-à-tête graph Γ , put a vertex in the middle of each hexagon A_j and connect them by edges through the center of every k_i , connecting the vertices of the adjacent hexagons. The tête-à-tête twist lengths on the two boundary annuli are 1 and 2, respectively.

For the case of D_n , n odd, take k_1 to be the spanning arc of b_1 and let $k_{i+1} = \varphi^i(k_1)$, $i = 1, \dots, 2n-3$. As before, we have $\varphi(k_{2n-2}) = k_1$, and

the k_i decompose S into $n - 1$ disks A_1, \dots, A_{n-1} , as in Figure 8. In Figure 10 (top), φ maps $A_1 \mapsto A_2 \mapsto \dots \mapsto A_{n-1}$ by right translations and sends A_{n-1} back to A_1 by a rotation of 180° .

If n is even, we use two orbits of intervals instead of one: define k_1, \dots, k_{n-1} and k'_1, \dots, k'_{n-1} by letting k_1, k'_1 be the spanning arcs of b_1, b_n respectively and $k_{i+1} = \varphi^i(k_1)$, $k'_{i+1} = \varphi^i(k'_1)$. Again, the union of the k_i and the k'_i decomposes S into disks A_1, \dots, A_{n-1} (see Figure 9). In Figure 10 (bottom), the monodromy maps $A_1 \mapsto A_2 \mapsto \dots \mapsto A_{n-1} \mapsto A_1$ by translations. The tête-à-tête graphs for D_n have one vertex at the center of each square and edges pass through the k_i and k'_i . Twist lengths on the boundary annuli are 1, $n - 2$ for odd n , and $1, 2, \frac{n}{2} - 1$ for even n .

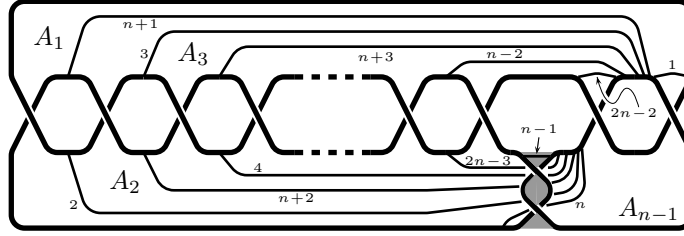


FIGURE 8. Decomposing arcs k_1, \dots, k_{2n-2} on the fibre surface of D_n for odd n .

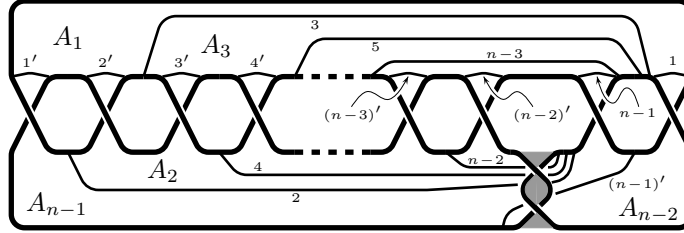


FIGURE 9. Decomposing arcs k_1, \dots, k_{n-1} , k'_1, \dots, k'_{n-1} on the fibre surface of D_n for even n .

4. THE FINITE CASES.

In [BIRS, Corollary 2], Buck et al. show that $T(2, n)$ admits only finitely many arcs preserving fibredness (up to isotopy). More precisely, they show that every clean arc is isotopic (free on the boundary) to an arc that is contained in one of the disks A_1, A_2 from the above

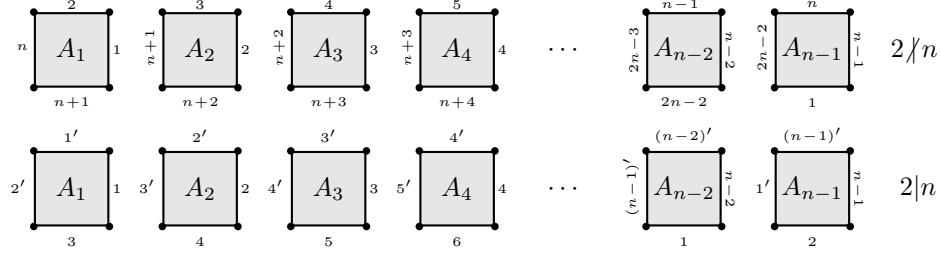


FIGURE 10. Description of the monodromy of D_n , for odd n (top) and for even n (bottom).

description of the monodromy of torus links. Apart from this infinite family of torus links, there are only three more torus links with just a finite number of arcs that preserve fibredness:

Proposition 1. *The torus links $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$ admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.*

For positive tree-like Hopf plumbed surfaces we similarly obtain:

Proposition 2. *The positive tree-like Hopf plumbings associated to any of the Coxeter-Dynkin trees A_n , D_n , E_6 , E_7 or E_8 admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.*

The proofs of Propositions 1 and 2 are rather technical and will be given in Section 6. Nevertheless, the idea is very simple: let S be the fibre surface of any torus link $T(n, m)$, given as thickening of a complete bipartite graph on $n+m$ vertices, or of D_n or E_7 , as described in Section 3. An arc $\alpha \subset S$ is determined up to isotopy by its endpoints and by the sequence of bands K it passes through. Now start listing all possible such sequences that yield clean arcs, for increasing length of the sequence. In order to prove finiteness of this list, we use three Lemmas, also given in Section 6. The intuitive meaning of Lemma 1 and Lemma 2 can be phrased as follows: if α and $\varphi(\alpha)$ intersect and this intersection seemingly cannot be removed by an isotopy, then α is indeed unclean. Lemma 3 asserts that a clean arc cannot stay in the complement of the graph for a distance of more than ℓ consecutive bands, where ℓ is the tête-à-tête twist length on the corresponding boundary annulus (for example, $\ell = 2$ for all torus links).

This is made precise in Section 6, using a notion of *arcs in normal position* (cf. Definition 3). Along with this case-by-case analysis, one can find all possible fibred links obtained from $A_{n-1} = T(2, n)$, $D_4 =$

$T(3, 3)$, D_n , $E_6 = T(3, 4)$, E_7 and $E_8 = T(3, 5)$ by cutting along an arc. Consult Table 1 for a complete list.

From	one obtains by cutting along a clean arc
$T(2, n)$	$T(2, n - 1)$, $T(2, m_1) \# T(2, m_2)$ for $m_1 + m_2 = n$
$T(3, 3)$	$T(2, 4)$, $(T(2, 2) \# T(2, 2) \# T(2, 2))^{*1}$
$T(3, 4)$	D_5 , $T(2, 6)$, $T(2, 5) \# T(2, 2)$, $T(2, 3) \# T(2, 3) \# T(2, 2)$, $(T(2, 3) \# T(2, 2) \# T(2, 3))^{*2}$
$T(3, 5)$	E_7 , D_7 , $T(2, 8)$, $(D_5 \# T(2, 3))^{*3}$, $T(2, 5) \# T(2, 4)$, $T(2, 7) \# T(2, 2)$, $T(3, 4) \# T(2, 2)$, $T(2, 5) \# T(2, 3) \# T(2, 2)$, $(T(2, 5) \# T(2, 2) \# T(2, 3))^{*4}$
D_n	$T(2, n)$, D_{n-1} , $D_{m_1} \# T(2, m_2)$ for $m_1 + m_2 = n$, $T(2, 2) \# T(2, 2) \# T(2, n - 2)$
E_7	E_6 , D_6 , $T(2, 7)$, $T(2, 4) \# T(2, 2) \# T(2, 3)$, $T(2, 6) \# T(2, 2)$, $T(2, 5) \# T(2, 3)$

$K_1 \# K_2$ denotes the connected sum of K_1 and K_2 , D_n denotes the closure of the braid $\sigma_1^{n-2} \sigma_2 \sigma_1^2 \sigma_2$, $n \geq 3$, and E_n denotes the closure of the braid $\sigma_1^{n-3} \sigma_2 \sigma_1^3 \sigma_2$, $n = 6, 7, 8$.

^{*1} chain of four successive unknots.

^{*2} both Hopf link components are summed to one trefoil knot each.

^{*3} both possible sums appear (trefoil summed with the unknot component of D_5 as well as trefoil summed with the trefoil component of D_5).

^{*4} one component of the Hopf link in the middle is summed to $T(2, 5)$ and the other is summed to the trefoil.

TABLE 1. Fibred links obtained from the exceptional torus links by cutting along an arc.

5. ARCS FOR LINKS WITH INFINITE ORDER MONODROMY

Theorem 3. *Let S be a surface obtained by iterated plumbing of positive Hopf bands and suppose the monodromy $\varphi : S \rightarrow S$ is pseudo-Anosov. Then, S contains infinitely many non-isotopic cutting arcs preserving fibredness.*

Proof. The monodromy φ is a composition of right Dehn twists along the core curves of the Hopf bands used for the construction of S as a Hopf plumbing. Let α be an arc dual to the core curve of the last plumbed Hopf band and such that α does not enter any of the previously plumbed Hopf bands. Then, in the product of Dehn twists representing φ , only the last factor affects α . It follows that α is clean

(and therefore $\varphi^n(\alpha)$ is also clean by Remark 1). Since φ is pseudo-Anosov and α is essential, the length of $\varphi^n(\alpha)$ (with respect to an auxiliary Riemannian metric) grows exponentially as n tends to infinity (compare [FM], Section 14.5). In particular, the arcs $\varphi^n(\alpha)$ are pairwise non-isotopic and clean. \square

In general, it is not enough to require φ to be non-periodic. Indeed, the family of arcs $\varphi^n(\alpha)$ might be finite, even if φ is of infinite order. This occurs typically when φ is reducible and α is contained in a periodic reducible piece of φ . However, if we dispose of a homology class $[c] \in H_1(S, \mathbb{Z})$ whose coordinate dual to α grows (i.e. $i(\varphi^n(c), \alpha) \rightarrow \infty$ for $n \rightarrow \infty$), then the family $\varphi^{-n}(\alpha)$ contains infinitely many distinct arcs since $i(\varphi^n(c), \alpha) = i(c, \varphi^{-n}(\alpha)) \rightarrow \infty$.

Proposition 3. *Let S be a surface obtained by plumbing positive Hopf bands according to a tree, other than A_n , D_n , E_6 , E_7 and E_8 . Then, S contains infinitely many non-isotopic cutting arcs preserving fibredness.*

Proof. Let S be a surface obtained by positive tree-like Hopf plumbing. Denote the induced action of the monodromy on $H_1(S, \mathbb{Z})$ by φ_* , and let $e_1, \dots, e_n \in H_1(S, \mathbb{Z})$ be the basis vectors represented by the core curves of the Hopf bands used in the plumbing construction. It follows from A'Campo's work on the spectrum of Coxeter transformations ([AC1]) and slalom knots ([AC2]), that φ_* has a real eigenvalue λ with $|\lambda| > 1$ if the tree corresponds to neither spherical nor affine Coxeter systems. Let c be an eigenvector of φ_* for the eigenvalue λ . Then the sequence $\varphi_*^n(c) = \lambda^n c$ is unbounded. Choose $j \in \{1, \dots, n\}$ such that the j -th coordinate of c is nonzero. Let $\alpha \subset S$ be a spanning arc of the Hopf band with core curve e_j . Then α is clean, since cutting along α yields a connected sum of positive tree-like Hopf plumbings, which is fibred. Moreover we have $|i(\varphi^n(c), \alpha)| \rightarrow \infty$ for $n \rightarrow \infty$ by construction. It therefore remains to study the affine Coxeter-Dynkin trees. For these, the spectral radius of φ_* is equal to one. However, φ_* has a Jordan block to the eigenvalue -1 in these cases, and a similar reasoning applies. \square

Proof of Theorem 2. Combine Propositions 2 and 3. \square

6. PROOF OF PROPOSITIONS 1 AND 2

Proposition 1. *The torus links $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$ admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.*

Proposition 2. *The positive tree-like Hopf plumbings associated to any of the Coxeter-Dynkin trees A_n , D_n , E_6 , E_7 or E_8 admit, up to*

isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.

Before we begin with the proofs, some notation and remarks are necessary. Let S be the fibre surface of either $T(n, m)$, D_n or E_7 , and let $\varphi : S \rightarrow S$ be the monodromy. Precisely as in Section 3, we decompose S into finitely many disjoint polygonal disks A_i (and B_j in the case of torus links) that are glued using bands (K_{ij} for the torus links and neighbourhoods of the k_i , k'_i for D_n and E_7). We use the letter D to denote any of the disks and the letter K to denote any of the bands. Let U be the union of all the disks, and let $N \subset S$ be the neighbourhood of the tête-à-tête graph on which φ is assumed to be periodic.

Definition 3. An arc $\alpha \subset S$ is in *normal position* if the following conditions hold:

- (a) The endpoints of α lie in ∂U .
- (b) For every band K , $\alpha \cap K \setminus U$ consists of finitely many straight segments parallel to the edges of the tête-à-tête graph.
- (c) The number of such segments in K is minimal among all arcs isotopic to α .
- (d) α intersects the graph transversely in finitely many points of U .
- (e) $\alpha \setminus N \subset U$, that is, before α enters N and after it leaves N , it stays in the disks that contain its endpoints.
- (f) $\alpha \cap U$ consists of finitely many straight arcs.

Remarks 2 (on normal position).

- Any arc can be brought into normal position by a free isotopy.
- If α is in normal position, then $\varphi(\alpha)$ can be brought into normal position keeping N fixed. Indeed, it suffices to straighten the two subarcs $\varphi(\alpha) \setminus N$ (or, undoing the twisting that occurs in the respective annuli), sliding the endpoints of $\varphi(\alpha)$ along ∂S , see Figure 11.

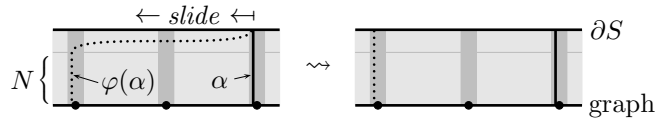


FIGURE 11. How to bring $\varphi(\alpha)$ in normal position, keeping N fixed.

- If α and $\varphi(\alpha)$ are in normal position as above, we may isotope $\varphi(\alpha)$ with endpoints fixed and keeping it in normal position, such that α and $\varphi(\alpha)$ intersect transversely in finitely many points of U . In particular, the sets $\alpha \setminus U$ and $\varphi(\alpha) \setminus U$ are now disjoint (cf. Figure 12).

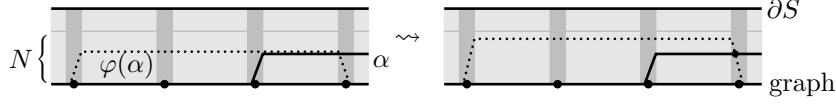


FIGURE 12. How to make $\alpha, \varphi(\alpha)$ intersect transversely, keeping normal position.

- Let α be in normal position and suppose it passes through at least one band. Let K be the first (respectively last) band traversed by α after (before) it starts (ends) at a boundary point p of one of the disks, say D . Then, p cannot lie between K and one of the two bands adjacent to K on ∂D . Otherwise, an isotopy sliding the starting point (endpoint) of α along ∂K would decrease the number of segments in K , contradicting part (c) of Definition 3.

Remarks 3 (compare the *bigon criterion*, Prop. 1.7 in [FM]). Suppose α intersects $\varphi(\alpha)$. If α is clean, there must be a bigon $\Delta \subset S$ whose sides consist of a subarc of α and a subarc of $\varphi(\alpha)$. If $\alpha, \varphi(\alpha)$ are in normal position, such Δ takes a particularly simple form:

- Δ cannot be contained in U (i.e., in one of the disks A_i or B_j). This would contradict part (f) of the above Definition 3.
- None of the two sides of Δ is contained in U , since the other side of Δ would have to leave U through one of the bands K and return through the same K . The disk Δ would then yield an isotopy reducing the number of segments of $\alpha \cap K$ or $\varphi(\alpha) \cap K$, contradicting part (c) of Definition 3.
- For every band K , $\Delta \cap K \setminus U$ consists of rectangles with two opposite sides parallel to the edge passing through K .
- $\Delta \cap U$ consists of topological disks δ connected to at least one rectangle.
- Construct a spine T for Δ as follows: put a vertex for each δ and connect two vertices by an edge if the corresponding disks δ connect to the same rectangle. T is a tree, for Δ is contractible. Two of its vertices correspond to the vertices of the bigon Δ . Among the other vertices of T , there is none of degree one because the adjacent edge would correspond to a rectangle in some K whose sides parallel to its core edge both belong to the same arc (α or $\varphi(\alpha)$). In other words, either α or $\varphi(\alpha)$ would pass through K and immediately return through K in the opposite direction. This would contradict part (c) of Definition 3. Therefore, T is a line consisting of some number of consecutive edges, and the two extremal vertices correspond to the vertices of Δ .

Lemma 1. *Let $\alpha, \varphi(\alpha)$ be in normal position and suppose they intersect in a point $p \in D$, where D is one of the disks A_i (or B_j in the torus link case). Let α', α'' be the components of $\alpha \cap D, \varphi(\alpha) \cap D$ containing p . If no two of the four points $\partial\alpha' \cup \partial\alpha'' \subset \partial D$ lie in the same band K , then α cannot be clean.*

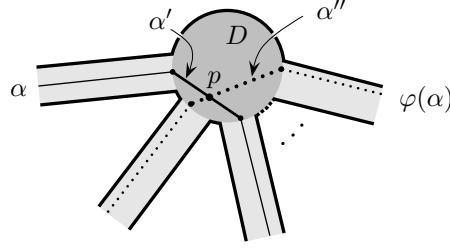


FIGURE 13. α cannot be clean by Lemma 1.

Remark 4. Note that we did not exclude the possibility that one of the endpoints of α or $\varphi(\alpha)$ lie in $\partial\alpha' \cup \partial\alpha''$.

Proof of Lemma 1. If α were clean, there would be a bigon. After possibly removing a certain number of such bigons, we are left with a bigon Δ with vertex p . By Remark 3, Δ has to leave D through one of the adjacent bands. Since one of the sides of Δ is a subarc of α and the other side is a subarc of $\varphi(\alpha)$, we find two points among $\partial\alpha' \cup \partial\alpha''$ that lie in this band, contradicting the assumption on α', α'' . \square

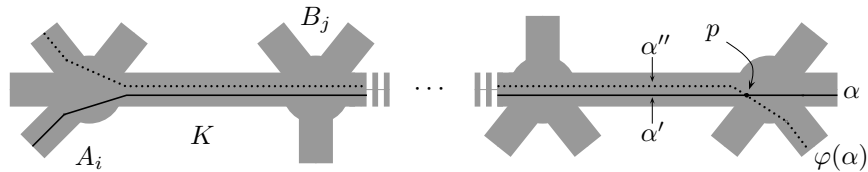


FIGURE 14. α cannot be clean by Lemma 2.

Lemma 2. *Let $\alpha, \varphi(\alpha)$ be in normal position and let α', α'' be subarcs of $\alpha, \varphi(\alpha)$ respectively (not necessarily contained in U). Suppose that the four endpoints of α' and α'' are contained in ∂U and that no two of them lie on the same band K . We further assume that α' and α'' intersect in exactly one point and that α', α'' run through the same bands (see Figure 14). Then α cannot be clean.*

Proof. Assume $\alpha' \cap \alpha'' = \{p\}$, then $p \in U$. As in the proof of Lemma 1, study a bigon Δ that starts at p . Δ consists of a sequence of rectangles as described in Remarks 3. Starting at p , it therefore has to pass through the same bands as α' and α'' . Since p was the only intersection between α' and α'' , Δ has to pass through at least one more band. But this is impossible by the assumption on the endpoints of α' and α'' . \square

Lemma 3. *A clean arc in normal position cannot traverse more than ℓ consecutive bands along a complementary annulus of twist length ℓ .*

Here, a sequence of bands $K^{(1)}, K^{(2)}, \dots$ is *consecutive*, if the set $(\bigcup_r K^{(r)} \cup \bigcup_i A_i \cup \bigcup_j B_j) \setminus \Gamma$ has a connected component that intersects all bands $K^{(r)}$ of the sequence in this order, i.e. it is possible to stay on the same side of the graph when walking along the bands. The twist length ℓ denotes the number of edges of Γ enclosed between γ and $\varphi(\gamma)$, where γ is a spanning arc of the corresponding boundary annulus that ends at a vertex of Γ (compare Section 3).

Proof. Suppose that α is a clean arc in normal position that traverses n consecutive bands, $n \geq \ell + 1$. We may assume that n is the maximal number of consecutively traversed bands. In these bands as well as the adjacent disks, isotope α such that it stays on one side of the graph, keeping it in normal position. Now bring $\varphi(\alpha)$ into normal position transverse to α as described in the Remarks 2. Recall the description of the monodromy φ as a tête-à-tête twist from Section 3: cutting the surface S open along the graph results in d annuli, where d is the number of components of $\partial S = L$ and each annulus has a link component as one boundary and a cycle consisting of edges of the graph as the other boundary. In one of these annuli we will see a subarc $\alpha' \subset \alpha$ that has exactly its endpoints in common with the graph and that travels near the edge boundary for a distance of n consecutive edges. (Note that α' cannot have any endpoint on ∂S . This would contradict part (c) of Definition 3). Let C be the disk bounded by α' and the graph. The monodromy φ keeps the link-boundary of this annulus fixed and rotates the neighbourhood N of the graph boundary by ℓ edges. Since $n \geq \ell + 1$, $\varphi(\alpha')$ has one of its endpoints in C and the other outside of C , so α' has to intersect its image $\varphi(\alpha')$ in a point $p \in U$, and we may assume that p is the only intersection between α' and $\varphi(\alpha')$. Denote by q the endpoint of $\varphi(\alpha')$ that lies in C and let D be the disk A_i or B_j containing q . Then make sure that $p \in D$ by an isotopy on $\varphi(\alpha')$ preserving normal position if necessary (compare Figures 15 and 16). However α is clean, so there must be a bigon in S whose sides consist of a subarc of α and a subarc of $\varphi(\alpha)$. After possibly

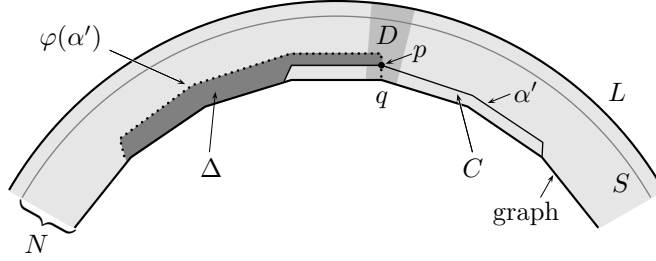


FIGURE 15. A normal arc passing through more than ℓ consecutive bands has to intersect its image under the monodromy (here $\ell = 2$). Part of an a priori possible bigon Δ .

removing a certain number of such bigons, we will be left with a bigon Δ starting at p . From the Remarks 3 we know that Δ has to leave D and consists of a sequence of rectangles. Let R be the first rectangle in this sequence, i.e. R is contained in a band adjacent to D . Let K^-, K^+ be the two bands adjacent to D that contain segments of α' , K^+ being the one that also contains a segment of $\varphi(\alpha')$ (see Figure 16). Let β be

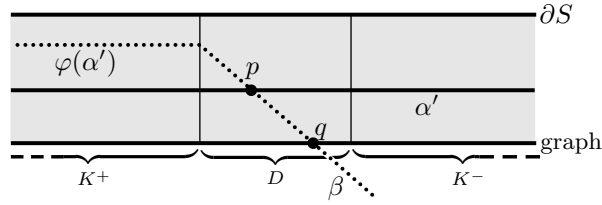


FIGURE 16. Part of the mentioned annulus, where the arcs α' and $\varphi(\alpha')$ intersect in a point $p \in D$.

the component of $\varphi(\alpha) \setminus \{p\}$ that contains q . We claim that β cannot leave D through K^- nor K^+ . Indeed, if β would leave D through K^- , $\varphi(\alpha)$ would traverse $n + 1$ consecutive bands, contradicting the assumption on n being maximal. On the other hand, if β would leave D through K^+ , we could reduce the number of segments in $\varphi(\alpha) \cap K^+$, contradicting the normal position of $\varphi(\alpha)$, i.e. part (c) of Definition 3. In contrast, α leaves D , starting from p in both directions, through K^- and K^+ . Consider now the subarcs of α and $\varphi(\alpha)$ that constitute two opposite sides of the rectangle R . Since R is contained in a band adjacent to D , these two subarcs arrive at ∂D through the same band, and they connect directly to $p \in D$. Therefore, we must have $R \subset K^+$, since K^+ is the only band containing two subarcs of α and $\varphi(\alpha)$ that directly connect to $p \in D$. Furthermore, R has to be the region enclosed

by $\alpha' \cap K^+$ and $\varphi(\alpha') \cap K^+$. Following $\varphi(\alpha')$ in the direction from q to p , we see that it leaves D through K^+ as one of the sides of R and continues staying on the same side of the graph for exactly $n - 1$ more edges. By assumption, p is the only intersection between α' and $\varphi(\alpha')$, so the bigon Δ has to continue for at least $n - 1$ more rectangles through consecutive bands. Similarly, the sides of these rectangles that are subarcs of α have to continue for at least $n - 1$ consecutive bands. We obtain a contradiction to the maximality of n , because α' ends after $n - \ell$ bands starting from D , since φ rotates the graph by ℓ edges. This finishes the proof. \square

Proof of Proposition 1. We will concentrate on the most complicated case of the torus knot $T(3, 5)$. It contains all difficulties appearing in the proofs for $T(3, 3)$ and $T(3, 4)$ which go along the same lines with fewer cases to consider. For each link appearing in Table 1 of Section 4, we will indicate one (but not every) possible choice of a cutting arc that yields the link in question. Let hence S be the fibre surface of $T(3, 5)$ and let $\alpha \subset S$ be any arc that preserves fibredness, i.e. a clean arc. Bring α into normal position using an isotopy (not fixing the boundary), cf. Remarks 2. Since φ permutes the vertices $\{a_i\}$ cyclically as well as the vertices $\{b_j\}$, it suffices to show that there are only finitely many clean arcs starting at a point of ∂A_1 or at a point of ∂B_1 , up to isotopy. We may further assume that α starts either at a point of ∂A_1 between k_{11} and k_{15} or at a point of ∂B_1 between k_{21} and k_{31} .

Case A. α starts at ∂A_1 , between k_{11} and k_{15} . Then, α cannot continue through either of the bands K_{11} nor K_{15} by the last item of Remarks 2. So, either α stays in A_1 (and there are only four such arcs up to isotopy), or it continues through K_{12}, K_{13} or K_{14} . If α stays in A_1 , the links obtained by cutting are E_7 (e.g. if α ends between k_{11} and k_{12}) and D_7 (e.g. if α ends between k_{12} and k_{13}).

Case A.1. α continues through K_{12} . Arriving in B_2 , there are three possibilities: either α ends at a point of ∂B_2 between k_{22} and k_{32} (and cutting along α yields $T(3, 4) \# T(2, 2)$), or it continues through K_{22} or K_{32} (ending at other points of ∂B_2 is impossible by the last item of Remarks 2).

Case A.1.1. α continues through K_{22} . Arriving in A_2 , α can end at a point of ∂A_2 (cutting yields $T(2, 7) \# T(2, 2)$ if α ends between k_{24} and k_{25} , and $T(2, 3)$ summed with the unknot component of D_5 if α ends between k_{23} and k_{24}), or it can continue through K_{23} or K_{24} . It cannot continue through K_{21} , since K_{12}, K_{22}, K_{21} is a sequence of three consecutive bands, so α would not be clean by Lemma 3. Finally, α

cannot continue through K_{25} . If it did, α and $\varphi(\alpha)$ would intersect in a point of A_1 , and Lemma 1 would imply that α cannot be clean (see Figure 17, top left). Note that we do not know whether the mentioned intersection is the only one since we do not know how α ends.

Case A.1.1.1. α continues through K_{23} . From B_3 , it cannot continue through K_{13} , for K_{22}, K_{23}, K_{13} are consecutive (Lemma 3). If it continues through K_{33} it cannot continue through any band adjacent to A_3 . Indeed, K_{23}, K_{33}, K_{32} are consecutive, so α cannot continue through K_{32} . If it would continue through K_{34} or K_{35} or K_{31} , we could apply Lemma 2 to the band K_{33} to show that α is not clean (see Figure 17).

Case A.1.1.2. α continues through K_{24} . If it ends in B_4 between k_{14} and k_{34} , we obtain $T(2, 5) \# T(2, 4)$ after cutting. Otherwise, it can continue from B_4 through K_{14} or through K_{34} .

Case A.1.1.2.1. If it continues through K_{14} , it cannot go further. Firstly, K_{24}, K_{14}, K_{15} are consecutive, so K_{15} is no option (Lemma 3). Neither can it proceed through K_{11} (this would produce a self-intersection of α) nor K_{12} (for otherwise we could apply Lemma 1 to an intersection between α and $\varphi(\alpha)$ in A_1). If it continues through K_{13} , it cannot go on through K_{23} since K_{14}, K_{13}, K_{23} are consecutive (Lemma 3). Suppose it continues through K_{33} . From A_3 , it cannot proceed through any of K_{31}, K_{35}, K_{34} , for otherwise we could apply Lemma 2 to the bands K_{13} and K_{33} , with an intersection between α and $\varphi(\alpha)$ occurring in A_3 (see Figure 17 left). However, α cannot continue through K_{32} either, because we could again apply Lemma 2, this time for the band K_{24} and an intersection in A_2 (see Figure 17 right).

Case A.1.1.2.2. α continues from B_4 through K_{34} . If it ends in A_3 between k_{35} and k_{31} , cutting yields $T(2, 5) \# T(2, 2) \# T(2, 3)$. Otherwise, it cannot continue from A_3 through K_{33} since K_{24}, K_{34}, K_{33} are consecutive. Neither can it proceed through K_{32} (apply Lemma 1 to A_3). So α can only continue through K_{35} or K_{31} .

Case A.1.1.2.2.1. If it continues through K_{35} , the only option to go further is through K_{15} , since K_{34}, K_{35}, K_{25} are consecutive. From A_1 (compare Figure 17), it cannot continue through K_{11} nor K_{12} (apply Lemma 2 to K_{15} with an intersection occurring in A_1). Neither can it continue through K_{14} , since K_{35}, K_{15}, K_{14} are consecutive. So it has to go through K_{13} . Arriving in B_3 , it cannot continue through K_{23} (apply Lemma 2 to K_{34} with an intersection occurring in B_4). Therefore α has to continue through K_{33} . From A_3 , it cannot proceed further. Firstly, K_{32} is not an option (otherwise apply Lemma 2 to K_{34} and K_{24} with an intersection in A_2). Neither can it go through K_{34} or K_{35} (apply Lemma 2 to K_{15}, K_{13}, K_{33} with an intersection occurring in A_3).

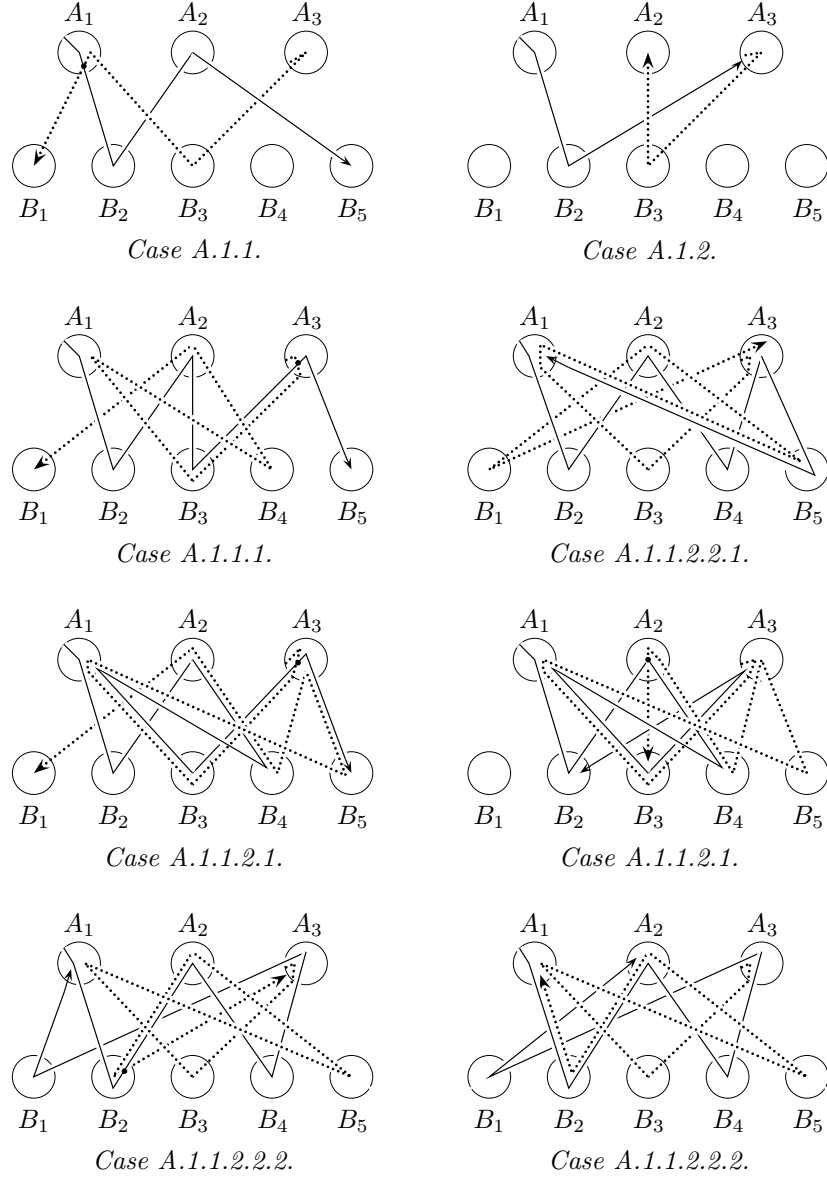


FIGURE 17. Schematic illustration for a selection of the cases in the proof of Proposition 1. The arc α is drawn as solid line, whereas $\varphi(\alpha)$ is shown as a dotted line.

Finally, it cannot pass through K_{31} either (apply Lemma 2 to the bands K_{15}, K_{13}, K_{33} with an intersection occurring in A_3).

Case A.1.1.2.2.2. If it continues through K_{31} and arrives in B_1 , it cannot proceed through K_{11} (apply Lemma 2 to K_{22} with an intersection occurring in B_2 , see Figure 17 left). So it has to go through K_{21} .

From A_2 , it cannot proceed through K_{22} , for K_{31}, K_{21}, K_{22} are consecutive. Neither can it go through either of K_{23} nor K_{24} (apply Lemma 1 to an intersection occurring in A_2 , see Figure 17 right). Finally, K_{25} can be ruled out by Lemma 2, applied to the bands K_{22} and K_{12} , with an intersection occurring in A_1 .

Case A.1.2. α continues through K_{32} (see Figure 17). Arriving in A_3 , it cannot continue through any band. Firstly, K_{12}, K_{32}, K_{33} are consecutive, so α cannot continue through K_{33} . If it would continue through any of the other bands adjacent to A_3 , α would intersect $\varphi(\alpha)$ in a point of A_3 such that we could apply Lemma 1 to obtain a contradiction to α being clean.

Case A.2. α proceeds through K_{13} . If it ends in B_3 between k_{23} and k_{33} , we obtain $T(2, 8)$ after cutting. From B_3 , it can continue through K_{23} or through K_{33} .

Case A.2.1. α continues through K_{23} . It cannot go on via K_{22} , for K_{13}, K_{23}, K_{22} are consecutive. Neither can it continue through K_{21} or K_{25} by Lemma 1 applied to an intersection in A_1 . If it next passes through K_{24} , it cannot go on through K_{14} , because K_{23}, K_{24}, K_{14} are consecutive. Proceeding through K_{34} , it can end in A_3 between k_{31} and k_{32} (this yields $T(2, 5) \# T(2, 3) \# T(2, 2)$). However, the only possibility for α to go further is via K_{32} , for K_{24}, K_{34}, K_{33} are consecutive (so K_{33} is no option), and α cannot continue through K_{35} nor K_{31} by applying Lemma 2 to the band K_{34} with an intersection of $\alpha, \varphi(\alpha)$ in A_3 . So α continues through K_{32} and arrives in B_2 . From there, it cannot continue through K_{12} (apply Lemma 2 to K_{22} and an intersection in B_3). If it continues through K_{22} , it cannot go further: K_{23} is impossible because K_{32}, K_{22}, K_{23} are consecutive, K_{24} can be excluded by Lemma 1, applied to A_2 , and K_{21} as well as K_{25} can be ruled out by Lemma 2, applied to K_{23} and K_{13} with an intersection occurring in A_1 .

Case A.2.2. α continues through K_{33} . This is similar to Case A.2.1. Again there is always a single option to go on, until there is no possibility left after four more steps.

Case A.3. α continues through K_{14} . This is analogous to Case A.1.

Case B. α starts at ∂B_1 between k_{21} and k_{31} . Then, it can only continue through K_{11} by the last item of Remarks 2. From A_1 , it can proceed through four distinct bands.

Case B.1. α continues through K_{15} . Since K_{11}, K_{15}, K_{25} are consecutive, it can a priori only continue through K_{35} . But this is impossible as well by Lemma 1, applied to the intersection between α and $\varphi(\alpha)$ occurring in B_1 .

Case B.2. α continues through K_{12} . This is analogous to Case B.1.

Case B.3. α continues through K_{14} . Arriving in B_4 , it can end between k_{24} and k_{34} (this results in $T(2, 3)$ summed with the trefoil component of D_5).

Case B.3.1. α continues through K_{24} . From A_2 , it cannot continue through K_{23} because K_{14}, K_{24}, K_{23} are consecutive (Lemma 3). Neither can it go on through K_{22} nor K_{21} (apply Lemma 1 to A_1). Suppose α continues through K_{25} . From B_5 , it cannot go on via K_{35} since K_{24}, K_{25}, K_{15} are consecutive. If it proceeds via K_{35} , we can apply Lemma 2 to the band K_{11} with an intersection in B_1 to obtain a contradiction.

Case B.3.2. α continues through K_{34} . From A_3 , there are only two options for α to proceed further. Indeed, K_{14}, K_{34}, K_{35} are consecutive, so K_{35} is out of the question. K_{31} can be ruled out by Lemma 1 for A_3 . The remaining possibilities are K_{32} and K_{33} .

Case B.3.2.1. α continues through K_{32} . From there, it cannot continue through K_{22} (apply Lemma 2 to K_{32}). So it has to branch off via K_{12} to A_1 . From there, it cannot continue through K_{15} since otherwise α would self intersect in A_1 . K_{11} is impossible as well, for K_{32}, K_{12}, K_{11} are consecutive. K_{15} can be ruled out using Lemma 1 for A_3 . So α can only continue through K_{13} , and from there only through K_{23} (K_{12}, K_{13}, K_{33} are consecutive). From A_2 , it cannot go on through any band except K_{25} . Indeed, K_{22} is impossible because K_{13}, K_{23}, K_{22} are consecutive. K_{21} and K_{24} can be ruled out by applying Lemma 2 to (K_{34}, K_{14}) and K_{23} respectively. After passing through K_{25} , α cannot go further: K_{15} is impossible by Lemma 2 (applied to K_{23}, K_{25}) and K_{35} can be ruled out by applying Lemma 2 to K_{34}, K_{14}, K_{11} .

Case B.3.2.2. α continues through K_{33} . Then, K_{13} cannot be next since K_{34}, K_{32}, K_{13} are consecutive. Thus α passes through K_{23} . From A_2 , it cannot go on via K_{24} , for K_{33}, K_{23}, K_{24} are consecutive. K_{21} and K_{22} are impossible as well (apply Lemma 2 to K_{14}). So α has to go through K_{25} . Then, it cannot proceed through K_{15} (apply Lemma 2 to K_{25}). It cannot go via K_{35} either (apply Lemma 2 to K_{14}, K_{11}), so α cannot continue at all.

Case B.4. α continues through K_{13} . This is analogous to Case B.3 and finishes the proof. \square

Proof of Proposition 2. We will present a case by case analysis for the possible clean arcs α in the fibre surface S of each of E_7 and D_n . The reader interested in studying the proof is advised to follow the arguments along with a pencil and copies of Figures 7 and 10, top and bottom. As in the proof of Proposition 1 above, we will make extensive use of Lemma 3 to exclude further polygon edges that α might

cross on its way from its starting point to its end. In order to keep the proof short, we will usually refer to such situations by just saying " α is trapped", or by saying that an edge "is a trap", meaning that α would traverse too many consecutive bands to be clean.

(E_7) First, let S be the fibre surface of E_7 , denote its monodromy φ and let $\alpha \subset S$ be a clean arc. Bring α into normal position with respect to k_1, \dots, k_9 . Note that the set of vertices of the hexagons A_1, A_2, A_3 decompose into two orbits under φ , namely the orbit of the vertex of A_1 between k_1 and k_2 , and the orbit of the vertex of A_1 between k_2 and k_7 . We may therefore assume by Remark 1 that α starts at one of these two vertices.

Case 1. α starts at the vertex of A_1 between k_1 and k_2 . Define an involution $\tau : S \rightarrow S$ as follows: τ interchanges hexagons A_1 and A_2 and then reflects A_1, A_2, A_3 along the diagonals parallel to k_7, k_8, k_1 respectively, whereby it induces the permutation $(13)(49)(58)(67)$ on the edges (k_1, \dots, k_9) . We have $\varphi \circ \tau \circ \varphi = \tau$, $\tau \circ \varphi$ fixes the vertex of A_1 between k_1 and k_2 and swaps the edges k_4, k_8 as well as the edges k_5, k_7 . By Remark 1, we may therefore assume that α either continues through k_4 or through k_5 .

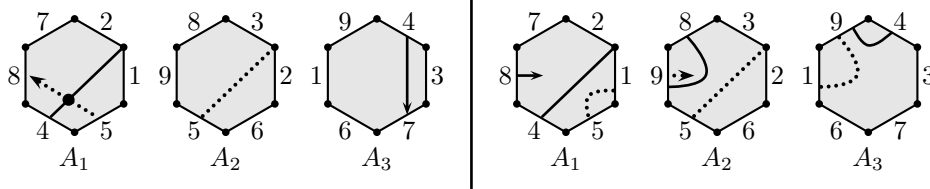


FIGURE 18. Illustration of two of the steps in Case 1.1.
The arc α is drawn as solid line, whereas $\varphi(\alpha)$ is shown as a dotted line.

Case 1.1. α continues through k_4 . From A_3 , it can only choose k_9 . Indeed, k_7, k_6 and k_1 would imply an intersection in A_1 (Lemma 1, compare Figure 18, left), and k_3 is consecutive to k_4 (Lemma 3, applied to the component with twist length one). Arriving in A_2 , k_2 and k_6 would imply an intersection in A_2 , so continuation is possible through k_3, k_5, k_8 only. But if α continues through k_5 or k_8 , it will be trapped (compare Figure 18, right). Therefore it goes through k_3 . Arriving in A_3 , it has to go through k_4 (k_9 implies an intersection in A_3 and k_1, k_6, k_7 imply intersections in A_1). However, passing through k_4 , α is trapped.

Case 1.2. α continues through k_5 . From A_2 , it can continue through k_3, k_6, k_8 or k_9 (k_2 implies an intersection in A_2). If it passes through k_6 or k_9 , it is trapped. So k_8 and k_3 are the only possibilities left.

Case 1.2.1. α continues through k_8 . Upon arrival in A_1 , it cannot continue through k_2, k_5 (intersection in A_2) nor through k_1 (this would imply an intersection in A_1). But if it continues through either of k_7 or k_4 , it is trapped.

Case 1.2.2. α continues through k_3 . From A_3 , α cannot go on through k_9, k_1 (this would imply an intersection in A_3). If it passes through k_4 , it is trapped. Suppose it continues through k_6 . Arriving in A_2 , it cannot continue through k_9, k_8, k_3 (this would produce an intersection in A_3), nor through k_2 (intersection in A_2). Finally, continuing through k_5 , it will be trapped. Therefore α has to continue from A_3 through k_7 . Arriving in A_1 , it can continue through k_2, k_8 or k_4 (k_1 implies an intersection in A_1 and k_5 implies an intersection in A_2). But all of these are traps.

Case 2. α starts at the vertex of A_1 between k_2 and k_7 . Define an involution $\sigma : S \rightarrow S$ as follows: σ interchanges A_1 and A_2 and then reflects A_1, A_2, A_3 along the diagonals parallel to k_1, k_2, k_4 respectively, inducing the permutation (19)(28)(37)(46) on the edges. As in Case 1, we have $\varphi \circ \sigma \circ \varphi = \sigma$, and $\sigma \circ \varphi$ fixes the vertex of A_1 between k_2 and k_7 , swapping k_4 and k_5 as well as k_1 and k_8 . By Remark 1, we may therefore assume that α continues through either k_1 or k_5 . However, if α continues through k_1 , it is trapped. Therefore it continues through k_5 . From A_2 , it can continue through k_8 or k_9 (k_6 is a trap and k_2, k_3 imply intersections in A_2).

Case 2.1. α continues through k_8 . From A_1 , it cannot continue through any of k_2, k_1, k_5 , because this would produce an intersection in A_2 , and k_7 is a trap. Therefore, it continues through k_4 and arrives in A_3 . Continuation through k_1 produces an intersection in A_3 , and k_6, k_7, k_3 imply intersections in A_1 . Finally, k_9 is a trap.

Case 2.2. α continues through k_9 . Arriving in A_3 , it can only continue through k_1 or k_4 (any other continuation produces an intersection in A_1). However, both k_1 and k_4 are traps, ending the proof for E_7 .

(D_n , n even) Now, suppose n is even and let α be a clean arc in the fibre surface S of D_n in normal position with respect to $k_1, \dots, k_{n-1}, k'_1, \dots, k'_{n-1}$. Define an involution $\tau : S \rightarrow S$ as follows: τ permutes the disks A_i according to the rule $\tau(A_i) = A_{n-i+2}$ for $i = 1, \dots, n-1$ and then reflects every A_i on the diagonal that contains the vertex between k_i and k_{i+2} (all indices are to be taken modulo n). Again $\varphi \circ \tau \circ \varphi = \tau$, and $\tau \circ \varphi$ fixes the vertex of A_1 between k'_1 and k'_2 as

well as the vertex between k_1 and k_3 , and swaps the other two vertices. We may therefore assume that α starts at a vertex of A_1 which is not the vertex between k'_2 and k_3 .

Case 1. α starts at the vertex of A_1 between k_1 and k'_1 . If it continues through k'_2 , it is already trapped. So it has to continue through k_3 . Arriving in A_3 , it can continue through k'_3 , k'_4 or k_5 .

Case 1.1. α continues from A_3 through k'_3 . From A_2 , it cannot continue through k_2 (otherwise it would intersect with $\varphi(\alpha)$), so it can only proceed through k'_2 or k_4 . However, both are traps.

Case 1.2. α continues from A_3 through k'_4 . This is similar to Case 1.1: arriving in A_4 , α can only continue through k'_5 (which is a trap) or k_4 . If it goes through k_4 , it has to continue from A_2 through k'_3 (k_2 produces an intersection in A_2 and k'_2 produces an intersection in A_3). Then however, it is trapped again.

Case 1.3. α can therefore continue from A_3 through k_5 only. In A_5 , the same situation reproduces, except that all indices in consideration are now shifted by $+2$. Therefore the only way for α to continue from A_5 is by passing through the edges k_7, k_9, k_{11}, \dots . After at most $n/2$ more steps, α will be trapped.

Case 2. α starts at the vertex of A_1 between k'_1 and k'_2 . Using τ again, we may assume that it continues through k_1 to A_{n-2} . If it goes through k'_{n-1} next, it is trapped since it is forced to follow the sequence of edges $k_{n-1}, k'_{n-2}, k_{n-2}, k'_{n-3}, \dots$. If it goes through k'_{n-2} to A_{n-3} instead, it can only continue from there through k'_{n-3} or k_{n-1} , and these are traps again. So it has to continue from A_{n-2} through k_{n-2} . In A_{n-4} , the same situation as one step earlier (where α arrived through k_1 in A_{n-2}) reproduces, except that all indices appearing in the consideration are now shifted by -2 . Hence the only way α can continue from A_{n-4} is by going through the sequence of edges $k_{n-4}, k_{n-6}, k_{n-8}, \dots$. After at most $n/2$ steps, α will be trapped.

Case 3. α starts at the vertex of A_1 between k_1 and k_3 . Using the involution τ from above, we may assume that it continues through k'_2 . From A_2 , it cannot go on through k_4 , for this would imply an intersection in A_2 . However, the two possibilities that remain (k'_3 and k_2) are traps, which ends the proof for D_n , n even.

(D_n , n odd) Finally, let n be odd and let S be the fibre surface of D_n . Suppose again we have a clean arc $\alpha \subset S$ in normal position with respect to k_1, \dots, k_{2n-2} . Since the monodromy permutes the A_i cyclically and since there are only two orbits of vertices of the A_i , we may assume that α starts in A_1 , at the vertex between k_1 and k_2 , or at the vertex between k_2 and k_n . As before, we then make use of

Remark 1 with the help of the involution $\tau : S \rightarrow S$ defined as follows: $\tau(A_i) = A_{n-i+2}$ by translations followed by a reflection on the diagonal of A_i that contains the vertex between k_{n+i-1} and k_{n+i} for $i = 1, 2$ and reflection on the diagonal of A_i that contains the vertex between k_i and k_{i+1} for $i = 3, \dots, n-1$. Applying Remark 1 as before, we may assume that α either starts at the vertex of A_1 between k_1, k_2 and continues through k_n (say), or that it starts at the vertex of A_1 between k_2 and k_n , continuing through k_1 (say). So there are two cases to consider, one being very similar to Case 1 above and the other similar to Case 3. No new arguments are needed. \square

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